Flows on the torus

Spencer Unger

University of Toronto

August 25, 2023

Joint work with Andrew Marks, and Anton Bernshteyn and Anush Tserunyan

Equidecomposition of functions

Definition

Let f, g be real valued functions on space X. f and g are *equidecomposible as functions* if there are functions $h_0, \ldots, h_m$ and transformations $T_1, \ldots T_m$ such that $f = h_0 + h_1 + \cdots + h_m$ and $g = h_0 + T_1 h_1 + \cdots + T_m h_m$.

Flows

Let $G = (X, E)$ be a directed symmetric graph on X. Let $\mathbb{R}^X$ be the set of real valued functions on X. And let Φ_E be the set of functions $\phi : E \to \mathbb{R}$ such that $\phi(x, y) = -\phi(y, x)$ for all $(x, y) \in E$.

Define an action of Φ_E on $X^{\mathbb{R}}$ by

$$(\phi \oplus f)(x) = f(x) - \sum_{y \in N(x)} \phi(x, y)$$

We say ϕ is an f-flow in G if $\phi \oplus f = 0$.

Equivalence

Let $T_1, \ldots T_m$ be transformations of X and let E be the symmetric graph generated by $T_1, \ldots T_m$ and their inverses.

Proposition

f and g are equidecomposible as functions using the translations $T_1, \ldots T_m$ if and only if there is an $f - g$-flow in (X, E).

By some algebra involving the action, we have $\phi \oplus (f - g) = 0$ if and only if $\phi \oplus f = g$.

Our setting

We work with bounded, real valued, Borel functions f on $\mathbb{T}^k$ and translations $T_1, \dots T_m$ and we attempt to find flows that are Borel.

Constructing flows

By the proposition, we change perspective and let $f : \mathbb{T}^k \to \mathbb{R}$ be a function with $\int f d\lambda = 0$ and try to construct an f-flow ϕ the graph $G = (X, E)$ generated by some fixed translations $T_1, \dots T_m$ and their inverses.

1. Define a sequence of functions $f_0 = f, f_1, \dots$ which are given by some iterated averaging procedure using the edges of G.
2. Associated to this averaging procedure, define a sequence of flows ϕ_n such that $\phi_n \oplus f_n = f_{n+1}$.
3. If we can show that $\lim_{n\to\infty} f_n = 0$ and $\phi = \sum_{i=0}^{\infty} \phi_i$ is absolutely convergent, then ϕ is an f-flow in G.

$$\phi \oplus f = \left(\sum_{i=0}^{\infty} \phi_i\right) \oplus f = \lim_{N\to\infty} \left(\sum_{i=0}^{N} \phi_i\right) \oplus f = \lim_{N\to\infty} f_{N+1} = 0$$

Averaging using random walks

(Joint work with Anton Bernshteyn and Anush Tserunyan)

Let G be the Schreier graph on $\mathbb{T}^k$ generated by the $\mathbb{Z}^d$ action of d randomly chosen translations *and the standard generators* $S = \{\pm e_1, \ldots, \pm e_d\}$.

For $h : \mathbb{T}^k \to \mathbb{R}$, let

$$\Delta h(x) = \frac{1}{2d} \sum_{e \in S} h(e \cdot x)$$

Note that if ϕ is the flow that assigns to each edge $(x, e \cdot x)$ the value $(1/2d)(h(e \cdot x) - h(x))$, then $\phi \oplus h = \Delta h$.

Wishful thinking

Let $f : \mathbb{T}^k \to \mathbb{R}$ be a function with $\int f d\lambda = 0$ and consider the sequence $f, \Delta f, \Delta^2 f, \dots$.

We hope the sequence goes to 0 and moreover that the sum of the flows implementing the sequence is absolutely convergent.

Define $\phi_n(x, e \cdot x) = \frac{1}{2d}\Big((\Delta^n f)(e \cdot x) - (\Delta^n f)(x)\Big)$ and compute the sum:

$$\phi(x, e \cdot x) = \sum_{n=0}^{\infty} \phi_n = \sum_{n=0}^{\infty} \frac{1}{2d}\Big((\Delta^n f)(e \cdot x) - (\Delta^n f)(x)\Big)$$

$$= \frac{1}{2d}\Big(\sum_{n=0}^{\infty}(\Delta^n f)(e \cdot x) - \sum_{n=0}^{\infty}(\Delta^n f)(x)\Big)$$

So we would really like that $\sum_{n=0}^{\infty}(\Delta^n f)(x)$ converges absolutely.

If it does, then we have $\phi \oplus f = 0$.

Discrepancy

We note that $(\Delta^n f)(x) = \int f d\rho_n^x$ where ρ_n^x is the probability measure defined on the orbit of x where $\rho_n^x(y)$ is the probability that a simple random walk starting at x ends at y after n steps.

For a measure μ and a function f, we define the *discrepancy* of f with respect to μ to be $D(f, \mu) = |\int f d\lambda - \int f d\mu|$. When f is the characteristic function of a set A, then we write $D(A, \mu)$.

So to show that $\sum_{n=0}^{\infty} \Delta^n f$ is absolutely convergent it is enough to bound $D(f, \rho_n^x)$ for all n and x.

From functions to intervals

The following lemma is due to Laczkovich:

Lemma

Let μ be a probability measure on $\mathbb{T}^k$. Suppose that $A \subseteq \mathbb{T}^k$ with $\mathrm{Dim}(\partial A) < k$, then there are $\epsilon \in (0,1)$ and constant K such that $D(A, \mu) \leq KD(\mu)^{\epsilon}$. Moreover, the ϵ and K only depend on $\mathrm{Dim}(\partial A)$ rather than A.

Where Dim is the upper Minkowski dimension and $D(\mu) = \sup D(B, \mu)$ where the sup is over axis parallel boxes.

From functions to intervals 2

We define a notion of dimension of the boundary of a function $\mathrm{Dim}(\partial f)$ as follows. Let

$$(\partial_\delta f)(x) = \sup_{y \in B_\delta(x)} f(y) - \inf_{y \in B_\delta(x)} f(y)$$

Then

$$\mathrm{Dim}(\partial f) = k + \lim_{\delta \to 0^+} \frac{\log \int \partial_\delta f d\lambda}{-\log(\delta)}$$

We prove an analog of the previous theorem for functions with small boundary.

Bounding $D(\rho_n^x)$

The previous argument allows us to reduce to bounding $D(\rho_n^x)$. Some ideas:

1. We use the Erdös-Turan-Koksma theorem from which it is enough to bound the Fourier coefficients of ρ_n^x.
2. These Fourier coefficients are independent of x!
3. Using the randomly chosen translations it is enough to bound the expected value of our sum.
4. Bounding the final sum uses the transience of the random walk in the d-dimensional grid for $d \geq 3$.

Generalization

Recall we found that our flow was easily definable from $g = \sum_{n=0}^{\infty} \Delta^n f$ and note that this satisfies the formula:

$$f = (1 - \Delta)g$$

This suggests that finding flows is similar to inverting operators like $1 - \Delta$ and leads to the following generalization:

Let $\mathscr{B}(\mathbb{T}^k)$ be the set of bounded Borel functions on $\mathbb{T}^k$.

Theorem (Bernshteyn-Tserunyan-U)

There is a natural class of operators $\mathscr{P}_d(\bar{u})$ on $\mathscr{B}(\mathbb{T}^k)$ defined uniformly from $\bar{u} \in (\mathbb{T}^k)^d$ such that the following holds. For almost every $\bar{u} \in (\mathbb{T}^k)^d$, for all $f \in \mathscr{B}(\mathbb{T}^k)$ with $\int f d\lambda = 0$ and for all $p \in \mathscr{P}_d(\bar{u})$, if $\mathrm{Dim}(\partial f)$ is sufficiently small relative to a fixed function of p and k, then there is $g \in \mathscr{B}(\mathbb{T}^k)$ such that $f = (1 - p)g$.

Corollary

If f, g *are bounded Borel functions on* $\mathbb{T}^k$ *with the same Lebesgue integral and* $\mathrm{Dim}(\partial f), \mathrm{Dim}(\partial g) < k$*, then* f *and* g *are equidecomposible as functions with Borel witnesses using translations.*

Another way of averaging

(Joint work with Andrew Marks)
Recall our framework for constructing flows of a function f was to define a sequence of functions f_n and "flows between them".

In the context of circle squaring, we can define $f_n(x)$ to be the average of f over an $|\cdot|_\infty$-norm box R of side length 2^n anchored at x in the action of $T_1, \dots, T_m$.

In this context it is straightforward to define flows ϕ_n such that $\phi_n \oplus f_n = f_{n+1}$.

Moreover, the absolute convergence of $\sum_{i=1}^{\infty} \phi_i$ is directly related to Laczkovich's discrepancy estimates.

Decomposing sets with small boundary

Laczkovich's 1992 theorem applies to sets $A, B \subseteq \mathbb{T}^k$ with the same positive Lebesgue measure and $\mathrm{Dim}(\partial A), \mathrm{Dim}(\partial B) < k$.

To bound the averages from the previous slide, $T_1, \ldots, T_m$ are chosen uniformly at random. In contrast, we have the following theorem:

Theorem (Marks-U)

Laczkovich's discrepancy estimates for sets of small boundary are possible using translations whose coordinates are either 0 or algebraic irrational.

Corollary

Known circle squaring results are possible with these vectors.

This answers a question of Laczkovich from his 1990 paper.

Some ideas from the proof

1. We make use of product actions.
2. We bound discrepancy of sets with small boundary with respect to product sets in terms of the one dimensional discrepancy of the components of the product.
3. The previous item allows us to replace the use of the Erdos-Turan-Koksma inequality with the Erdos-Turan inequality.
4. We use a theorem of Schmidt about simultaneous approximation of return times of quadratic irrationals.